

# Isochronous Spacetimes

\*<sup>♡</sup>Fabio Briscese<sup>1</sup> and <sup>+◇</sup>Francesco Calogero<sup>2</sup>

\*Istituto Nazionale di Alta Matematica Francesco Severi, Gruppo Nazionale di Fisica Matematica, Città Universitaria, P.le A. Moro 5, 00185 Rome, Italy

♡Dipartimento SBAI, Sezione di Matematica, University of Rome “La Sapienza”, via Antonio Scarpa 16, 00161 Roma, Italy

<sup>+</sup>Physics Department, University of Rome “La Sapienza”, p. Aldo Moro, I-00185 ROMA, Italy

<sup>◇</sup>Istituto Nazionale di Fisica Nucleare, Sezione di Roma

<sup>1</sup>fabio.briscese@sbai.uniroma1.it, briscese.phys@gmail.com

<sup>2</sup>francesco.calogero@roma1.infn.it, francesco.calogero@uniroma1.it

## Abstract

The possibility has been recently demonstrated to manufacture (nonrelativistic, Hamiltonian) many-body problems which feature an *isochronous* time evolution with an arbitrarily assigned period  $T$  yet mimic with good approximation, or even exactly, any given many-body problem (within a quite large class, encompassing most of nonrelativistic physics) over times  $\tilde{T}$  which may also be arbitrarily large (but of course such that  $\tilde{T} < T$ ). In this paper we review and further explore the possibility to extend this finding to a general relativity context, so that it becomes relevant for cosmology.

## 1 Introduction

In this paper we revisit the findings reported in [1] and report some additional considerations relevant for a better understanding of the validity—in the context of theoretical and mathematical physics—of those results.

It has been recently shown [2, 3] that—given a general autonomous dynamical system  $D$ , other autonomous dynamical systems  $\tilde{D}$  can be manufactured, featuring two additional *arbitrary* positive parameters  $T$  and  $\tilde{T}$  with  $T > \tilde{T}$  and possibly also two additional dynamical variables—which are characterized by the following two properties:

(i) For the same variables of the original dynamical system  $D$  the new dynamical system  $\tilde{D}$  yields, over the time interval  $\tilde{T}$ , hence over an *arbitrarily long* time, a dynamical evolution which mimics *arbitrarily closely* that yielded by the original system  $D$ ; up to corrections of order  $t/\tilde{T}$ , or possibly even *identically*.

(ii) The system  $\tilde{D}$  is *isochronous*: for arbitrary initial data all its solutions are completely periodic with a period  $T$ , which can also be arbitrarily assigned, except for the (obviously necessary) condition  $T > \tilde{T}$ .

Moreover it has been shown [2, 3] that, if the dynamical system  $D$  is a many-body problem characterized by a (standard, autonomous) Hamiltonian  $H$  which is translation-invariant (i. e., it features no external forces), other (also autonomous) Hamiltonians  $\tilde{H}$  characterizing *modified* many-body problems can be manufactured which feature the *same* dynamical variables as  $H$  (i. e., in this case there is no need to introduce two additional dynamical variables) and which yield a time evolution *quite close*, or even *identical*, to that yielded by the original Hamiltonian  $H$  over the *arbitrarily assigned* time  $\tilde{T}$ , while being *isochronous* with the *arbitrarily assigned* period  $T$ , of course with  $T > \tilde{T}$ .

Let us emphasize that the class of Hamiltonians  $H$  for which this result is valid is quite general. In particular it includes the standard Hamiltonian system describing an *arbitrary* number  $N$  of point particles with *arbitrary* masses moving in a space of *arbitrary* dimensions  $d$  and interacting among themselves via potentials depending *arbitrarily* from the interparticle distances (including the possibility of multiparticle forces), being therefore generally valid for any realistic many-body problem, hence encompassing most of nonrelativistic physics. This result is moreover true, *mutatis mutandis*, in a quantal context.

For instance let us tersely review here the case of the standard Hamiltonian describing the many-body problem, but focusing—merely for notational simplicity—on the case with equal particles (and setting their mass to unity) and a one-dimensional setting, which reads (in self-evident standard notation)

$$H(\underline{p}, \underline{q}) = \frac{1}{2} \sum_{n=1}^N (p_n^2) + V(\underline{q}) \quad (1a)$$

with the potential being translation-invariant,  $V(\underline{q} + a) = V(\underline{q})$  (i. e., no external forces) but otherwise unrestricted (except for the condition—again, for simplicity—that the time-evolution yielded by this Hamiltonian

be nonsingular). The standard approach to treat this problem is to introduce the center-of-mass coordinate  $Q(\underline{q}) = \frac{1}{N} \sum_{n=1}^N q_n$  and the total momentum  $P(\underline{p}) = \sum_{n=1}^N p_n$ , and to then focus on the coordinate and momenta in the center of mass,  $x_n = q_n - Q(\underline{q})$ ,  $y_n = p_n - P(\underline{p})/N$ , which characterize the physics of the problem and whose evolution is determined by the Hamiltonian  $h(\underline{y}, \underline{x})$  defined as follows:

$$H(\underline{p}, \underline{q}) = \frac{[P(\underline{p})]^2}{2N} + h(\underline{y}, \underline{x}) , \quad (1b)$$

$$h(\underline{y}, \underline{x}) = \frac{1}{2} \sum_{n=1}^N (y_n^2) + V(\underline{x}) , \quad (1c)$$

where  $V(\underline{x}) = V(\underline{q})$  thanks to the assumed translation-invariance of  $V(\underline{q})$ . An *isochronous* Hamiltonian  $\tilde{H}(\underline{p}, \underline{q}; T)$  reads then as follows,

$$\tilde{H}(\underline{p}, \underline{q}; T) = \frac{1}{2} \sum_{n=1}^N \left\{ [P(\underline{p}) + h(\underline{y}, \underline{x})]^2 + \left( \frac{2\pi}{T} \right) [Q(\underline{q})]^2 \right\} . \quad (2a)$$

It can indeed be shown [2, 3] that it entails an *isochronous* evolution (with period  $T$ ) of the center of mass coordinate  $Q$ , of the total momentum  $P$ , and—most importantly—of all the particle coordinates, whose evolution then reads

$$\tilde{\underline{x}}(t) = \underline{x}(\tau(t)) , \quad \tilde{\underline{y}}(t) = \underline{y}(\tau(t)) , \quad (2b)$$

where we indicate as  $\tilde{\underline{x}}(t)$ ,  $\tilde{\underline{y}}(t)$  the time evolution yielded by the tilded Hamiltonian  $\tilde{H}(\underline{p}, \underline{q}; T)$  and as  $\underline{x}(t)$ ,  $\underline{y}(t)$  the time evolution yielded by the original Hamiltonian  $H(\underline{p}, \underline{q})$ , and

$$\tau(t) = A \sin \left( \frac{2\pi t}{T} + \Phi \right) \quad (2c)$$

where  $A$  and  $\Phi$  are *constant* parameters given by simple expressions in terms of the initial values,  $Q(0)$  and  $P(0)$ , of the position of the center of mass of the system and of its total momentum, and of the Hamiltonian  $\tilde{H}(\underline{p}, \underline{q}; T)$  (which is of course a constant of motion for this time evolution). Note that, for  $|t| < T$ ,

$$\tau(t) = \alpha + \beta t + O \left[ \left( \frac{t}{T} \right)^2 \right] \quad \text{with} \quad \alpha = A \sin(\Phi) , \quad \beta = \frac{2\pi A}{T} \cos(\Phi) ; \quad (2d)$$

hence up to small corrections (*arbitrarily small* for an *arbitrarily large* assignment of the period  $T$  as long as  $|t|$  is in an assigned interval  $\tilde{T}$ , of course such that  $\tilde{T} \ll T$ ) the time evolution of the tilded coordinates approximates that of the untilded coordinates, up to a constant shift and rescaling of time.

Note that in this case—again, for the sake of simplicity—the modified, *isochronous* Hamiltonian  $\tilde{H}(\underline{p}, \underline{q}; T)$  features only one additional constant parameter,  $T$ ; but it has been shown [2, 3] that an analogous treatment—involving a somewhat more complicated definition of a modified, *isochronous* Hamiltonian, featuring then two arbitrarily assigned parameters,  $T$  and  $\tilde{T} < T$ —allows to manufacture a modified Hamiltonian that reproduces *exactly* (up to a constant rescaling and shift of the time variable) the same time evolution yielded by the original Hamiltonian over the interval  $\tilde{T}$  but is *isochronous* with period  $T$ —and let us again emphasize that this can be done for a many-body problem involving an *arbitrary* number of (possibly different) particles, moving in a space of *arbitrary* dimension  $d$  and interacting via *arbitrary* interparticle forces. Which therefore includes most of (nonrelativistic) physics.

Since it is impossible to distinguish experimentally dynamical systems that behave *arbitrarily closely*, or even *identically*, over an *arbitrarily long* period of time, this finding has various remarkable implications in the context of theoretical and mathematical physics. In particular it raises [2, 3] interesting questions about the distinction between integrable and nonintegrable evolutions, about the definition of chaotic behavior, and—for macroscopic systems featuring, say, a number of particles of the order of Avogadro's number—about statistical mechanics and the second principle of thermodynamics. And for, say,  $10^{85}$  particles it seems to have some cosmological relevance.

But the proper setting of cosmological theories is general relativity. For this reason in our previous paper [1] we investigated whether and how this result could be extended to that context. We concluded that this is possible provided one extends general relativity by allowing *degenerate* (i.e. non invertible) metrics. In particular we have shown [1] that for any homogeneous, isotropic and spatially flat metric  $g_{\mu\nu}$  satisfying

Einstein's equations and providing a model of the universe, it is possible to find a different (also homogeneous, isotropic and spatially flat) metric solution  $\tilde{g}_{\mu\nu}$  which is *locally* (in time) diffeomorphic to  $g_{\mu\nu}$ —hence yields the same cosmology as  $g_{\mu\nu}$  for any observation over an *arbitrary* time interval  $\tilde{T}$ —and is cyclic<sup>1</sup> (in fact periodic with an *arbitrary* period  $T > \tilde{T}$  in the time coordinate  $t$ ); but it is *degenerate* at an infinite, discrete sequence of times  $t_n = t_0 \pm nT/2$ ,  $n = 0, 1, 2, \dots$ . We interpreted these metrics as corresponding to *isochronous cosmologies* [1].

Let us also recall [1] that this result is not restricted to homogeneous, isotropic and spatially flat metrics: it can be easily extended to any synchronous metric, therefore it is quite general since most metrics can be written in synchronous form by a diffeomorphic change of coordinates.

Due to the diffeomorphic correspondence between  $g_{\mu\nu}$  and  $\tilde{g}_{\mu\nu}$  *locally* in time—for time intervals of order  $\tilde{T}$ —these two metrics give the same physics locally in time, and therefore there is no way to distinguish them using observations local in time; which is essentially the same finding valid in the context of the Hamiltonian systems considered in [2, 3] and tersely recalled above. In particular, the metric  $\tilde{g}_{\mu\nu}$ —while being cyclic on time scales larger than  $\tilde{T}$ —may yield over the time interval  $\tilde{T}$  just the standard sequence of domination firstly by radiation, then by matter and dark energy, as well as the cosmological perturbations consistent with all observational tests [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15], which characterize the metric  $g_{\mu\nu}$  of the standard  $\Lambda$ -CDM cosmological model (see for instance [16] for a review). Furthermore, it has been shown that such isochronous metrics  $\tilde{g}_{\mu\nu}$  can be manufactured to be geodesically complete [1] and therefore singularity-free<sup>2</sup>, so that the geodesic motion as well as all physical quantities described by scalar invariants are always well defined [1] and the Big Bang singularity may be avoided—even when some of the phenomenological observations can nevertheless be interpreted as remnants of a past Big Bang, which however may never be actually attained by the metric  $\tilde{g}_{\mu\nu}$ , neither in the past nor in the future. This implies that these isochronous metrics  $\tilde{g}_{\mu\nu}$  may describe a singularity-free universe which—while featuring a time evolution which reproduces identically (up to diffeomorphic time reparameterization; *locally* in time, except at a discrete set of instants  $t_n$ ) the standard cosmological model characterized by the metric  $g_{\mu\nu}$ —features an expansion which stops at some instant, to be followed by a period of contraction, until this phase of evolution stops and is again followed by an expanding phase, this pattern being repeated *ad infinitum*. Let us re-emphasize that the universe characterized by the metric  $\tilde{g}_{\mu\nu}$  may thereby avoid to experience the Big Bang singularity, even if the universe which it mimics locally in time, characterized by the metric  $g_{\mu\nu}$ , does encounter that singularity. Let us also note that this class of metrics realize *de facto* the reversal of time's arrow, as discussed for instance in [19].

But in order to do so, these isochronous metrics must be *degenerate* at a discrete set of instants  $t_n$  when the time reversals occur, say from expansion to contraction and viceversa; a phenomenology whose inclusion in general relativity might be considered problematic, because at these instants  $t_n$ —when the metric is *degenerate*—the "equivalence principle" corresponding to the requirement that the metric tensor have a Minkowskian signature is violated. On the other hand a *physically unobservable* violation of a "principle" can be hardly considered *physically relevant*.

Purpose and scope of this paper is to better clarify the meaning and the interpretation of the *isochronous spacetimes* introduced in [1], and to discuss some additional technical aspects. As stated above, these isochronous metrics have the property to be *degenerate* at an infinite set of times  $t_n$ , i.e. over an infinite number of 3-dimensional hypersurfaces, where generally the scale factor reaches its maximum and minimum values and the spacetime changes from an expanding to a contracting phase and vice versa [1].

The inclusion of degenerate metrics in general relativity is not a trivial matter; indeed, it is forbidden if it were required—as an absolute (as it were, "metaphysical") rule—that general relativity be consistent with the "equivalence principle", i. e. that spacetime *always* be a Pseudo-Riemannian manifold *with Minkowskian signature*. This would indeed imply that the solutions of Einstein's general relativity, to be acceptable, must be *nondegenerate* metric tensors, since wherever in spacetime a metric is *degenerate* it is not possible to define its inverse, hence the Christoffel symbols as well as the Riemann, Ricci and Einstein tensors are not defined there. Hence—so the argument of some critics goes—our isochronous metrics, which are not Minkowskian (locally, on the hypersurfaces  $t = t_n$ ) should be discarded in general relativity because they violate the equivalence principle at those degeneracy surfaces. Our rejoinder is that the metrics we introduced [1] are solutions of a version of general relativity whose dynamics, while still governed by Einstein's equations<sup>3</sup>, does allow the violation of the equivalence principle at a discrete set of hypersurfaces. It can be argued that such theories are therefore *different* from standard general relativity, if validity of the equivalence principle—enforced as an

<sup>1</sup>We prefer to talk of cyclic instead of periodic solutions due to the fact that periodicity is not an invariant concept, since it is not invariant under a redefinition of time. For a review of cyclic models see [4].

<sup>2</sup>A spacetime is singularity-free if it is geodesically complete, i.e. if its geodesics can be always past- and future-extended [17, 18].

<sup>3</sup>Einstein's equations by itself do not determine the signature of spacetime, see the discussion in [20, 21].

absolute, universal rule—is considered an essential element of that theory; although the difference is in fact *physically unobservable*. This situation is indeed quite analogous with the findings described above, valid in the nonrelativistic context of Hamiltonian many-body problems (classical or quantal), where the solutions of a physical theory described by a Hamiltonian  $H$  are arbitrarily well approximated or even identically reproduced over an arbitrary time interval by periodic solutions of a *different* Hamiltonian  $\tilde{H}$ , i.e. by a *different* physical theory. The main difference is that in the general relativistic context in the two *different* physical theories the dynamic is given by the *same* Einstein's equations, only the class of acceptable solutions is *different*.

The consideration in the context of general relativity of degenerate metrics is in any case not a novelty. For instance they were already introduced [21, 22], and subsequently investigated in a series of papers [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 46, 37, 38, 39], in order to investigate signature-changing spacetimes. This class of signature-changing metrics corresponds to a classical realization of the change of signature in quantum cosmology conjectured by Hartle and Hawking [40] (see also [41, 42, 43]), which has philosophical implications on the origin of the universe [44].

Below we will also consider a different realization of isochronous cosmologies via non-degenerate metrics featuring a jump in their first derivatives at the inversion times  $t_n$ , which then implies a distributional contribution in the stress-energy tensor at  $t_n$ . These metrics are diffeomorphic to the continuous and degenerate isochronous metrics everywhere except on the inversion hypersurfaces  $t = t_n$ , hence they describe the same physics except at the infinite set of discrete times  $t_n$ ; therefore they may also be manufactured so as to agree with all cosmological observations (*locally* in time). Since these two realizations of isochronous cosmologies are *locally* but *not globally* diffeomorphic, they correspond to different spacetimes and may be considered to emerge from *different* theories: the nondegenerate ones are generalized (in the sense of distributions) solutions of Einsteinian general relativity (including universal validity of the equivalence principle, but allowing jumps—whose physical significance and justification is moot—at the discrete times  $t_n$ ); the degenerate ones feature no mathematical or physical pathologies but fail to satisfy the “equivalence principle” at the discrete times  $t_n$ . They are essentially indistinguishable—among themselves and from non-isochronous cosmologies—by feasible experiments (unless one considers feasible an experiment lasting an arbitrarily long time...).

## 2 Isochronous Cosmologies

The procedure used to construct, for an autonomous dynamical system  $D$ , another autonomous dynamical system  $\tilde{D}$  whose solutions *approximate* arbitrarily closely, or even reproduce exactly, those of the system  $D$  over an *arbitrary* time interval  $\tilde{T}$  but are *isochronous* with an *arbitrary* period  $T > \tilde{T}$ , is based on the introduction of an auxiliary variable  $\tau$  in place of the physical time variable  $t$  [2, 3]. This procedure formally entails a change in the time dependence of any physical quantity  $f(t)$ , so that  $f(t)$  gets replaced by  $f(\tau(t))$  with  $\tau(t)$  a periodic function of  $t$  (with period  $T$ :  $\tau(t \pm T) = \tau(t)$ ), hence the new dynamics entails that all physical quantities evolve periodically with period  $T$ . This change, in the case of a nonrelativistic dynamical system, implies a change in the dynamical equations: for instance, in the case of the (quite general) many-body problem described above, the theory is characterized by a new (autonomous) Hamiltonian  $\tilde{H}$ —different from the original (autonomous) Hamiltonian  $H$ —which causes any physical variable originally evolving as  $f(t)$  under the dynamics yielded by  $H$ , to evolve instead as  $f(\tau(t))$  under the dynamics yielded by  $\tilde{H}$ . This change of the physical laws determining the time evolution of the system produces the *isochronous* evolution [3, 2]. But let us stress—as we already did in [1]—that any attempt to attribute to the new variable  $\tau$  the significance of “time” would be improper and confusing: the variable playing the role of “time” is the same,  $t$ , for both the original dynamical system  $D$  and the modified dynamical system  $\tilde{D}$ ; in particular, for that characterized by the standard many-body Hamiltonian  $H$ , see (1a), as well as for that characterized by the modified many-body Hamiltonian  $\tilde{H}$ , see (2a).

The dynamical systems  $D$  and  $\tilde{D}$  mentioned above feature the same trajectories in phase space, but while the time evolution of the dynamical system  $D$  corresponds to a *uniform* forward motion along those trajectories, the time evolutions of the modified dynamical systems  $\tilde{D}$ —although produced by time-independent equations of motion—correspond to a *periodic* (with assigned period  $T$ ), forward and backward, time evolution along those same trajectories, exploring therefore only a portion of them; with the possibility to manufacture  $\tilde{D}$  in such a way that, for an *arbitrary* subinterval  $\tilde{T} < T$ , the dynamics of  $D$  and  $\tilde{D}$  be *very close* or even *identical*.

In the context of general relativity, an analogous phenomenology exists [1]. However, in this case the situation is a bit different, due to the physical invariance of Einstein's equations under diffeomorphisms, which implies that the procedure used to construct the isochronous solutions, while again formally entailing the introduction of a new auxiliary variable  $\tau(t)$ , does not imply any change of the fundamental Einstein equations, corresponding instead to the identification of an enlarged class of solutions of these equations. But again any

attempt to attribute *globally* to this auxiliary variable  $\tau$  the significance of *time* would be improper and confusing; while by attributing to it *locally* (in time) the significance of *reparameterized time*, it is immediately seen that the enlarged class of metrics entails dynamical evolutions which, *locally in time*, are *physically equivalent* to that yielded by the original, unmodified metric.

Our point of departure is to assume a homogeneous, isotropic and spatially flat metric  $\tilde{g}_{\mu\nu}$  which, in a given reference frame  $(t, \vec{x})$ , is defined by the line element

$$ds^2 = b(t)^2 dt^2 - a(t)^2 d\vec{x}^2, \quad (3)$$

where  $b(t)^2$  is commonly termed the "lapse" function (although we did not use this term in [1]).

The lapse function is not determined by Einstein's equations; its introduction in (3) corresponds to the freedom to reparametrize time, which in general relativity is permissible, via a diffeomorphic transformation, with no physical effect: indeed, in general relativity it is the *geometry* of spacetime that characterizes the physics, and this geometry is not affected by the coordinates used to describe it. An additional requirement which is generally considered essential—amounting to the "equivalence principle"—is that the metric have a *Lorentzian signature* (everywhere in spacetime). This implies that, in the case of the metric (3), the lapse function  $b(t)$  must never vanish, so that one can always put this metric in its synchronous form via the reparameterization of the time variable from  $t$  to  $\tau$  such that

$$d\tau = b(t) dt, \quad \tau(t) = B(t) \equiv \int_0^t b(t') dt', \quad (4)$$

which maps the metric (3) into the Friedmann-Robertson-Walker metric  $g_{\mu\nu}$  with line element

$$ds^2 = d\tau^2 - \alpha(\tau)^2 d\vec{x}^2, \quad (5a)$$

where

$$\alpha(\tau(t)) \equiv a(t). \quad (5b)$$

As stated above, our trick to construct *isochronous* dynamical systems is *formally* based on the eventual replacement of the physical time variable  $t$  with another variable  $\tau$ . In a general relativistic context this corresponds to have a *periodic* lapse function (with an *arbitrarily assigned* period  $T$ ) having moreover a vanishing mean value, so that its integral  $B(t)$ , see (4), is also periodic with period  $T$ :

$$b(t+T) = b(t); \quad \tau(t+T) = \tau(t). \quad (6)$$

The choice (6)—when inserted in (3)—implies the following consequences [1].

(i) For any FRW metric (5a), the corresponding—via the relation (6)—metric (3) is periodic in the time variable  $t$ , and therefore it is cyclic, defining thereby an *isochronous spacetime*.

(ii) Since the lapse function is periodic with period  $T$  and has zero mean value, it must vanish at an infinite set of times  $t = t_n \equiv t_0 \pm nT/2$  with  $n = 0, 1, 2, 3, \dots$ , implying that the metric (3) is degenerate at these times.

(iii) If the metric (5a) is a solution of Einstein's equations (namely, of the FRW equations) corresponding to the stress energy tensor of a perfect fluid (or a mixture of different perfect fluids)  $T_{\mu\nu} = [r(\tau) + p(\tau)] u_\mu u_\nu - g_{\mu\nu} p(\tau)$ , where  $r(\tau)$  and  $p(\tau)$  are the energy density and pressure of the fluid and  $u_\mu = \delta_{\mu 0}$  is its 4-velocity, then (3) is also a solution of Einstein's equations corresponding to a stress energy tensor  $T_{\mu\nu} = [\rho(t) + P(t)] U_\mu U_\nu - g_{\mu\nu} P(t)$  of a perfect fluid with energy density  $\rho(t) = r(\tau(t))$ , pressure  $P(t) = p(\tau(t))$  and 4-velocity  $U_\mu = |b(t)|\delta_{\mu 0}$ . Therefore the metrics (3) and (5a) describe two universes filled with the same perfect fluid(s).

(iv) Since for  $t \neq t_n$  the two metrics (3) and (5a) are connected by a diffeomorphic transformation, they give the same physics in any time interval within which  $b(t)$  does *not* vanish, so that they are in fact experimentally indistinguishable there.

(v) One can conveniently choose the lapse function in such a way that the scale factor is always positive, i.e.  $a(t) = \alpha(\tau(t)) > 0$  for any  $t$ . This choice implies that all the scalar invariants as well as the pressure and energy density of the fluid remain finite at any  $t$  and the spacetime is geodesically complete, thus singularity free.

(vi) Geodesics have, in spacetime, a helical structure.

These solutions of course violate the "equivalence principle" at the infinite set of discrete times  $t_n$ . It is thus seen that, with the choice (6), the class of spacetime solutions of the equations of general relativity is enlarged by allowing unobservable violations of the "equivalence principle" at a discrete, numerable set of times  $t_n$ . This more general class of spacetimes do not seem to entail any *experimentally observable* differences; although they might entail a quite different structure of spacetime, for instance absence versus presence of Bing Bang singularities.

The problem of finding a generalization of general relativity which includes degenerate metrics has been studied in the past, motivated by the consideration of a class of metrics, the so called signature-changing metrics [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 46, 37, 38, 39], which give a classical realization of the change of signature in quantum cosmology conjectured by Hartle and Hawking [40, 41, 42, 43, 44]. The classical change of signature for a homogeneous, isotropic and spatially flat universe is realized by a metric tensor defined by the following line element

$$ds^2 = N(t) dt^2 - R(t)^2 d\vec{x}^2, \quad (7)$$

where  $N(t)$  is a continuous function changing sign at some time  $t_0$ , for instance  $N(t)$  is positive for  $t > t_0$ , negative for  $t < t_0$  and zero for  $t = t_0$  in correspondence to the hypersurface of change of signature where the metric (7) is degenerate (note the difference from (3), where the coefficient  $b^2(t)$  of  $dt^2$  might vanish at some points but never becomes *negative*,  $b^2(t) \geq 0$ ).

These generalizations of standard general relativity including degenerate metrics as (3) and (7) are not unique since they depend on their defining prescriptions: *different* prescriptions yield *different* physical theories [24]. In particular, it has been pointed out that different generalizations based on different prescriptions give different junction conditions on the degeneracy hypersurfaces, see for instance the discussion in [23, 24]. In fact "one can also consider discontinuities in the extrinsic curvature" [23] on the degeneracy hypersurface of a degenerate solution of Einstein's equations, "but attempts to relate these to a distributional matter source at the boundary require some form of field equations valid on the surface", see discussion in [24], and this would require some special prescription on what Einstein's equations are on the degeneracy hypersurfaces. Thus, if one accepts degenerate solutions of Einstein's equations as physically acceptable spacetimes, there is no reason to infer that a discontinuity of the extrinsic curvature on a degeneracy surface implies that the stress energy tensor associated to such a metric tensor must have a distributional form there. What is more, delta functions with support on degeneracy hypersurfaces have no significant effect in covariant integrals [36, 39], due to the fact that the covariant volume element  $dx^4 \sqrt{-g}$  associated to a degenerate metric is null on its degeneracy hypersurface, because the determinant  $g$  vanishes there. For instance, for the metric (3) one has, for any integrable function  $f(x)$ ,

$$\int dx^4 \sqrt{-g} \delta(t - t_n) f(x) = \int dt dx^3 |b(t)| |a(t)|^3 \delta(t - t_n) f(x) = |b(t_n)| |a(t_n)|^3 \int dx^3 f(x) = 0 \quad (8)$$

(because  $b(t_n) = 0$ ). Hence delta-like distributions centered on the degeneracy hypersurfaces  $t = t_n$  disappear from integrals (see also the discussion in [36]).

So, the question of how to include degenerate metrics in general relativity is moot.

In order to formulate a generalized gravitational theory which includes such metrics, it is convenient to focus directly on the Einstein's equations characterizing the geometry of spacetime, which themselves admit degenerate metrics as their solutions. If, however, one considers desirable to derive this theory from a variational principle, a possibility is to assume the standard Einstein-Hilbert gravitational action of general relativity, such that the total action of gravitation plus the matter (nongravitational) fields is

$$S_{Tot} = \int_M \sqrt{-g} \left( -\frac{1}{2k} R + L_{Mat} \right) d^4x, \quad (9)$$

where  $k = 8\pi G/c^4$ ,  $G$  is the gravitational constant,  $c$  the speed of light,  $M$  is the manifold defining the spacetime and  $L_{Mat}$  is the Lagrangian density of matter fields; and add the requirement that, for a metric tensor which is degenerate on some hypersurface  $\Sigma$ , the variation of the metric  $g_{\mu\nu}$  vanish on  $\Sigma$ , i.e.  $\delta g_{\mu\nu}|_{\Sigma} = 0$ . With such an assumption the variation of the action (9) is

$$\delta S_{Tot} = \frac{1}{2} \int_{M/\Sigma} \sqrt{-g} \left( -\frac{1}{k} G^{\mu\nu} + T^{\mu\nu} \right) d^4x, \quad (10)$$

where  $G^{\mu\nu}$  is the Einstein tensor and  $T^{\mu\nu}$  is the stress-energy tensor of matter fields, which gives the standard equations of general relativity in the region where the metric tensor is nondegenerate.

To bypass the problems related to degenerate signature-changing metrics, it has been proposed [21, 22] to consider a different, nondegenerate but discontinuous, realization of the classical change of signature, as given by the metric

$$ds^2 = f(\tau) d\tau^2 - R(\tau)^2 d\vec{x}^2, \quad (11)$$

with a discontinuous lapse function  $f(\tau) = 1$  for  $\tau > \tau_0$  and  $f(\tau) = -1$  for  $\tau < \tau_0$ . It has been shown [21, 22] that in this case it is possible to introduce a smooth (generalized) orthonormal reference frame which allows a

variational derivation of Einstein's equations and an extension of the Darmois formalism [46] to discontinuous metrics, in such a way that, also in the case of a discontinuous metric tensor, the discontinuity of the extrinsic curvature on a hypersurface is related to a distributional stress energy tensor [36]. This is achieved by adding a surface term to the Einstein-Hilbert action, in such a way that the gravitational action becomes

$$S_g = \int_{M/\Sigma} \sqrt{-g} R d^4x + \oint_{\Sigma} \sqrt{-g} K d\Sigma \quad (12)$$

where  $M$  is the manifold defining the spacetime,  $\Sigma$  is the boundary of  $M$  where the metric tensor is discontinuous,  $R$  is the Ricci scalar curvature and  $K$  the extrinsic curvature of  $\Sigma$ . The addition of such a boundary term is always necessary when one considers manifolds with boundaries [45]. Incidentally we note that this action cannot be used in the case of degenerate metrics, because in this case the degeneracy surface  $\Sigma$  has no unitary normal vector hence the extrinsic curvature  $K$  does not exist.

The two realizations of the classical change of signature, the continuous and degenerate one, see (7), and the discontinuous one, see (11), are *locally but not globally* diffeomorphic, since they are related by a change of the time variable  $dt = d\tau/\sqrt{|N(t)|}$  which is not defined at the time of signature change  $t_0$  when  $N(t) = 0$ . Hence they represent two different spacetimes. However, except for the instants when  $N(t)$  vanishes, they are *locally* diffeomorphic and thus they describe—excepts at those instants—the same physics.

As in the case of signature-changing metrics, it is also possible to consider a realization of *isochronous* cosmologies different from that we introduced in [1] and reviewed above (see (3) and the subsequent treatment), via *nondegenerate* metrics featuring a finite jump of their first derivatives at an infinite, discrete set of equispaced times. Such a realization is given by the following metric:

$$ds^2 = d\eta^2 - \tilde{a}(\eta)^2 d\vec{x}^2 \quad (13)$$

with

$$\alpha(\tau(\eta)) \equiv \tilde{a}(\eta) \quad (14)$$

and

$$d\tau = C(\eta) d\eta, \quad (15)$$

with a periodic function  $C(\eta)$  of period  $T$  given for instance by  $C(\eta) = 1$  for  $nT < \eta < (2n+1)T/2$  and  $C(\eta) = -1$  for  $(2n+1)T/2 < \eta < (n+1)T$  and  $n$  integer, hence such that

$$\tau(\eta) \equiv \int_0^\eta C(\eta') d\eta' = \eta - nT \quad \text{for } nT < \eta < (2n+1)T/2, \quad (16a)$$

$$\tau(\eta) \equiv \int_0^\eta C(\eta') d\eta' = (n+1)T - \eta \quad \text{for } (2n+1)T/2 < \eta < (n+1)T. \quad (16b)$$

The metric (13) is periodic with period  $T$  and is not degenerate at the inversion times  $\eta_n = nT/2$ , where its first derivatives feature a finite jump. Since (13) is nondegenerate, now the Darmois formalism [46] applies and one can relate the discontinuity of the first derivatives of the metric, hence the discontinuity of the extrinsic curvature of the hypersurfaces  $\Sigma_{\eta_n}$  defined by  $\eta = \eta_n$  (which is now well defined), with a distributional stress-energy tensor. In fact defining the unitary normal vector  $n_\alpha = \delta_{\alpha 0}$ , the tangent vectors  $e_\alpha^a = \delta_\alpha^a$  and the induced metric  $h_{ab} = -\tilde{a}(\eta)^2 \delta_{ab}$  of a hypersurface  $\Sigma_\eta$  of equation  $\eta = \text{constant}$ , which also implies  $n^\alpha = \delta^{\alpha 0}$ ,  $e_\alpha^a = \delta_\alpha^a$  and  $h^{ab} = -\delta_{ab}/\tilde{a}(\eta)^2$ , the extrinsic curvature of  $\Sigma_\eta$  is given by [17]

$$K_{ab} \equiv n_{\alpha;\beta} e_\alpha^a e_b^\beta = \alpha(\tau(\eta)) \alpha'(\tau(\eta)) C(\eta) \delta_{ab}, \quad (17)$$

where  $\alpha'(z) \equiv d\alpha(z)/dz$ <sup>4</sup>.

Therefore the discontinuity of the extrinsic curvature on the hypersurface  $\Sigma_{\eta_n}$  is

$$[K_{ab}]|_{\Sigma_{\eta_n}} = 2(-1)^n \alpha(\tau(\eta)) \alpha'(\tau(\eta)) \delta_{ab}, \quad (18)$$

and this implies that the stress-energy tensor has, on the numerable set of hypersurfaces  $\Sigma_{\eta_n}$ , a distributional contribution given by

$$T_{\alpha\beta}^{distr} = \sum_n \delta(\eta - \eta_n) S_{ab}|_{\Sigma_{\eta_n}} e_\alpha^a e_\beta^b, \quad (19)$$

---

<sup>4</sup>Note that in [1] we have calculated the extrinsic curvature of the hypersurfaces  $t = \text{const}$  of the metric (3) defining the normal vector of such hypersurfaces as  $n^\alpha = \delta^\alpha/b(t)$ , the tangent vectors as  $e_\alpha^a = a(t)\delta_\alpha^a$  and  $h_{ab} = -\delta_{ab}$ . Instead, (18) corresponds to the choice  $n^\alpha = \delta^{\alpha 0}/|b(t)|$ ,  $e_\alpha^a = \delta_\alpha^a$  and  $h_{ab} = -a(t)^2 \delta_{ab}$ .

where

$$S_{ab}|_{\Sigma_{\eta_n}} \equiv \frac{1}{8\pi} ([K_{ab}]|_{\Sigma_{\eta_n}} - [K]|_{\Sigma_{\eta_n}} h_{ab}) = \frac{(-1)^{n+1}}{2\pi} \alpha(\tau(\eta)) \alpha'(\tau(\eta)) \delta_{ab} \quad (20)$$

and  $[K]|_{\Sigma_{\eta_n}} \equiv [K_{ab}]|_{\Sigma_{\eta_n}} h^{ab}$ . We note that this distributional stress-energy tensor is absent in the case of a vacuum Minkowskian universe, since in this case  $\alpha(\tau)$  is constant and its derivative vanishes.

Again, let us point out that the two metrics (3) and (13) are two *different* realizations of an *isochronous* cosmology, since they are *locally*, but *not globally*, diffeomorphic via the change of variable  $d\eta = b(t)dt$  which is singular at  $t_n$ . In particular the statements (i,iii,iv,v,vi), see above, which are valid for the metric (3), also apply to the metric (13). Moreover, both (3) and (13) are *locally* (but *not globally*) diffeomorphic to the FRW metric (5a) and therefore they give an *identical* dynamics in any time interval such that  $b(t) \neq 0$  or  $C(\eta) \neq 0$ . The main difference is that the isochronous evolution given by (3) entails the introduction of degenerate metrics and the nondegenerate realization of the isochronous dynamics given by (13) implies the existence of a distributional contribution to the stress-energy tensor. In both cases the evolution of the universe is locally the same as that described by the FRW equations, implying that it is not possible to discriminate by experiments local in time between an isochronous and a noncyclic evolution of the universe.

### 3 Conclusions

In this paper we have reviewed and further discussed the idea introduced in [1], itself being an extension to a general relativity context—hence relevant to discuss cosmological models—of a finding valid in the wide context of nonrelativistic Hamiltonian dynamics. Let us emphasize that—as in that context—the essence of our observation is to point out the disturbing possibility, given a physical theory that describes reality, to manufacture variations of it—i. e., different theories, characterized by the introduction of an arbitrary parameter  $T$ —which describe essentially the *same* reality over time intervals smaller than  $T$  but are *cyclic* with period  $T$ .

And let us conclude this paper by emphasizing that we are *not* ourselves arguing that our Universe evolves *cyclically* indeed *isochronously*, much less wish we to enter the related philosophical issues (not our cup of tea!); we have merely pointed out the (disturbing!) fact that, in the context of theoretical and mathematical physics, such a possibility does not seem to be excluded; nor does it seem feasible to exclude it by realizable experiments.

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### References

- [1] F. Briscese and F. Calogero, *Isochronous cosmologies*, Int. J. Geom. Meth. Mod. Phys. **11**, 1450054 (2014) (19 pages), arXiv:1402.0704 [gr-qc].
- [2] F. Calogero and F. Leyvraz, *General technique to produce isochronous Hamiltonians*, J. Phys. A.: Math. Theor. **40**, 12931-12944 (2007).
- [3] F. Calogero and F. Leyvraz, *How to extend any dynamical system so that it becomes isochronous, asymptotically isochronous or multi-periodic*, J. Nonlinear Math. Phys. **16**, 311-338 (2009); F. Calogero and F. Leyvraz, *Isochronous systems, the arrow of time, and the definition of deterministic chaos*, Lett. Math. Phys. **96**, 37-52 (2011); F. Calogero, *Isochronous systems*, Oxford University Press, 2008 (marginally updated paperback edition, 2012); F. Calogero, *Isochronous dynamical system*, Phil. Trans. R. Soc. **A 369**, 1118-1136 (2011).
- [4] P. J. Steinhardt and N. Turok, *Cosmic Evolution in a Cyclic Universe*, Phys. Rev. **D 65**, 126003 (2002) [arXiv:hep-th/0111098]; P. J. Steinhardt and N. Turok, *A cyclic model of the universe*, Science **296**, 1436 (2002); J. Khoury et al., *Designing Cyclic Universe Models*, Phys. Rev. Lett. **92**, 031302 (2004) [arXiv:hep-th/0307132]; P. J. Steinhardt and N. Turok, *Why the cosmological constant is small and positive*, Science **312**, 1180 (2006) [astro-ph/0605173]; K. Saaidi et al., *Interacting New Agegraphic Dark Energy*



- in a *Cyclic Universe*, *Astrophys. Space Sci.* **338**, 355 (2012) [arXiv:1201.0275]; S. Nojiri et al., *Cyclic, ekpyrotic and little rip universe in modified gravity*, *AIP Conf. Proc.* **1458**, 207 (2011) [arXiv:1108.0767]; Y. F. Cai and E. N. Saridakis, *Non-singular Cyclic Cosmology without Phantom Menace*, *J. Cosmol.* **17**, 7238 (2011) [arXiv:1108.6052]; V. Sahni and A. Toporensky, *Cosmological Hysteresis and the Cyclic Universe*, *Phys. Rev. D* **85**, 123542 (2012) [arXiv:1203.0395]; P. Creminelli and L. Senatore, *A smooth bouncing cosmology with scale invariant spectrum*, *JCAP* **0711**, 010 (2007) [arXiv:hep-th/0702165]; Y. S. Piao, *Proliferation in Cycle*, *Phys. Lett. B* **677**, 1 (2009) [arXiv:0901.2644]; J. Zhang et al., *Amplification of Curvature Perturbations in Cyclic Cosmology*, *Phys. Rev. D* **82**, 123505 (2010) [arXiv:1007.2498]; Z. G. Liu and Y. S. Piao, *Scalar Perturbations Through Cycles*, *Phys. Rev. D* **86**, 083510 (2012) [arXiv:1201.1371]; J. Khoury et al., *The Ekpyrotic Universe: Colliding Branes and the Origin of the Hot Big Bang*, *Phys. Rev. D* **64**, 123522 (2001) [arXiv:hep-th/0103239]; R. Y. Donagi et al., *Visible Branes with Negative Tension in Heterotic M-Theory*, *JHEP* **0111**, 041 (2011) [arXiv:hep-th/0105199]; J. Khoury et al., *Density Perturbations in the Ekpyrotic Scenario*, *Phys. Rev. D* **66**, 046005 (2002) [arXiv:hep-th/0109050]; Y. Shtanov and V. Sahni, *Bouncing Braneworlds*, *Phys. Lett. B* **557**, 1 (2003) [arXiv:gr-qc/0208047]; T. Biswas et al., *Bouncing Universes in String-inspired Gravity*, *JCAP* **0603**, 009 (2006) [arXiv:hep-th/0508194]; K. Bamba et al., *Periodic Cosmological Evolutions of Equation of State for Dark Energy*, *Entropy* **2012**, 14(11), 2351-2374 [arXiv:1203.4226]; M. Novello, S.E.P. Bergliaffa, *Bouncing Cosmologies*, *Phys. Rep.* **463**, 127-213 (2008) [arXiv:0802.1634].
- [5] E. Komatsu et al. [WMAP collaboration], *Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation*, *Astrophys. J. Suppl.* **192**: 18 (2011) [arXiv:1001.4538].
- [6] B. Jain and A. Taylor, *Cross-correlation Tomography: Measuring Dark Energy Evolution with Weak Lensing*, *Phys. Rev. Lett.* **91**, 141302 (2003) [arXiv:astro-ph/0306046].
- [7] D. J. Eisenstein et al. [SDSS Collaboration], *Detection of the Baryon Acoustic Peak in the Large-Scale Correlation Function of SDSS Luminous Red Galaxies*, *Astrophys. J.* **633**, 560 (2005) [arXiv:astro-ph/0501171].
- [8] M. Tegmark et al. [SDSS Collaboration], *Cosmological parameters from SDSS and WMAP*, *Phys. Rev. D* **69**, 103501 (2004) [arXiv:astro-ph/0310723]; U. Seljak et al. [SDSS Collaboration], *Cosmological parameter analysis including SDSS Ly-alpha forest and galaxy bias: constraints on the primordial spectrum of fluctuations, neutrino mass, and dark energy*, *Phys. Rev. D* **71**, 103515 (2005) [arXiv:astro-ph/0407372].
- [9] S. Perlmutter et al. [SNCP Collaboration], *Measurements of Omega and Lambda from 42 High-Redshift Supernovae*, *Astrophys. J.* **517**, 565 (1999) [arXiv:astro-ph/9812133]; A. G. Riess et al. [Supernova Search Team Collaboration], *Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant*, *Astron. J.* **116**, 1009 (1998) [arXiv:astro-ph/9805201].
- [10] Planck Collaboration: P. A. R. Ade et al., *Planck 2013 results. XVI. Cosmological parameters*, [arXiv:1303.5076].
- [11] S. Capozziello et al., *Cosmographic Constraints and Cosmic Fluids*, *Galaxies* **2013**, **1**(3), 216-260 (2013) [arXiv:1312.1825].
- [12] A. Aviles et al., *Cosmography and constraints on the equation of state of the Universe in various parametrizations*, *Phys. Rev. D* **86**, 123516 (2012) [arXiv:1204.2007].
- [13] E. J. Copeland et al., *Dynamics of dark energy*, *Int. J. Mod. Phys. D* **15**:1753-1936 (2006) [arXiv:hep-th/0603057].
- [14] K. Bamba et al., *Dark energy cosmology: the equivalent description via different theoretical models and cosmography tests*, *Astrophys. Space Sci.* **342**: 155-228(2012) [arXiv:1205.3421].
- [15] BICEP2 I: *Detection of B-mode Polarization at degree angular scales*, BICEP2 Collaboration, P. A. R. Ade et al., arXiv:1403.3985v2.
- [16] V. Mukhanov, *Physical Foundations of Cosmology*, Cambridge University Press, 2005.
- [17] E. Poisson, *An Advanced Course in General Relativity*, University of Guelph (2002) (<http://www.physics.uoguelph.ca/~poisson/research/agr.pdf>).
- [18] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, Freeman & Co. (1973).

- [19] A. D. Sakharov, *Cosmological models of the Universe with reversal of time's arrow*, Zh. Eksp. Teor. Fiz. **79**, 689-693 (1980) [Sov. Phys. JETP **52**, 349-351 (1980)].
- [20] C. Teitelboim, *Hamiltonian structure of spacetimes, General Relativity and Gravitation: One Hundred Years after the Birth of Albert Einstein*, ed A. Held (New York; Plenum), 23-97 (1980).
- [21] G. Ellis, A. Sumerukt, D. Coulet, C. Hellaby, *Change of signature in classical relativity*, Class. Quantum Grav. **9** (1992) 1535-1554.
- [22] G. Ellis, *Covariant Change of Signature in Classical Relativity*, Gen. Rel. Grav. **24**, No. 10, 1992.
- [23] M. Carfora, G. Ellis, *The Geometry of Classical Change of Signature*, Intl. J. Mod. Phys. **D 4**, 175 (1995).
- [24] T. Dray, G. Ellis, C. Hellaby, *Note on Signature Change and Colombeau Theory*, Gen. Rel. Grav. **33** (2001) 1041-1046.
- [25] R. Mansouri, K. Nozari, *A New Distributional Approach to Signature Change*, Gen. Rel. Grav. **32**, 253-269 (2000).
- [26] T. Dray, C. A. Manogue, R. W. Tucker, *Particle Production from Signature Change*, Gen. Rel. Grav. **23**, 967 (1991).
- [27] T. Dray, C. A. Manogue, R. W. Tucker, *The Scalar Field Equation in the Presence of Signature Change*, Phys. Rev. D **48**, 2587 (1993).
- [28] C. Hellaby, T. Dray, *Failure of standard conservation laws at a classical change of signature*, Phys. Rev. D **49**, 5096-5104 (1994).
- [29] S. A. Hayward, *Weak Solutions Across a Change of Signature*, Class. Quantum Grav. **11**, L87 (1994).
- [30] T. Dray, C. A. Manogue, R. W. Tucker, *Boundary Conditions for the Scalar Field in the Presence of Signature Change*, Class. Quantum Grav. **12**, 2767-2777 (1995).
- [31] S. A. Hayward, *Comment on "Failure of Standard Conservation Laws at a Classical Change of Signature"*, Phys. Rev. D **52**, 7331-7332 (1995).
- [32] C. Hellaby and T. Dray, *Reply Comment: Comparison of Approaches to Classical Signature Change*, Phys. Rev. D **52**, 7333-7339 (1995).
- [33] S. A. Hayward, *Signature Change in General Relativity*, Class. Quant. Grav. **9**, 1851 (1992); erratum: Class. Quantum Grav. **9**, 2543 (1992).
- [34] M. Kossowski and M. Kriele, *Smooth and Discontinuous Signature Type Change in General Relativity*, Class. Quantum Grav. **10**, 2363 (1993).
- [35] T. Dray, *Einstein's Equations in the Presence of Signature Change*, J. Math. Phys. **37**, 5627-5636 (1996).
- [36] T. Dray, G. Ellis, C. Hellaby, C. A. Manogue, *Gravity and Signature Change*, Gen. Rel. Grav. **29**, 591-597 (1997).
- [37] W. Kamleh, *Signature Changing Space-times and the New Generalised Functions*, gr-qc/0004057.
- [38] T. Dray, C. Hellaby, *Comment on 'Smooth and Discontinuous Signature Type Change in General Relativity'*, Gen. Rel. Grav. **28**, 1401-1408 (1996).
- [39] D. Hartley, R. W. Tucker, P. A. Tuckey, T. Dray, *Tensor distributions on signature changing space-times*, Gen. Rel. Grav. **32** (2000) 491-503, gr-qc/9701046.
- [40] J. B. Hartle, S. W. Hawking, *Wave function of the universe*, Phys. Rev. **D 28**, 2960 (1983).
- [41] B. S. De Witt, *Quantum Theory of Gravity. I. The Canonical Theory*, Phys. Rev. **160**, 1113 (1967).
- [42] S. W. Hawking, in *Astrophysical Cosmology*, H. A. Bruck, G. V. Coyne and M. S. Longair, editors (Pontifica Academia Scientiarum, Vatican City, 1982), pp. 563-574.
- [43] S. W. Hawking, *The Quantum State of the Universe*, Nucl. Phys. B **239**, 257 (1984).

- [44] S. W. Hawking, *A Brief History of Time*, Banlam, London, 1989.
- [45] S. W. Hawking, Gary. T. Horowitz, *The Gravitational Hamiltonian, Action, Entropy, and Surface Terms*, Class. Quant. Grav. **13** (1996) 1487-1498, arXiv:gr-qc/9501014.
- [46] G. Darmais, *Memorial des Sciences Mathematiques*, Fascicule **25**, Gauthier-Villars, Paris, 1927; see also W. Israel, *Singular Hypersurfaces and Thin Shells in General Relativity*, Nuovo Cimento **44 B**, 1-14 (1966); Erratum, Nuovo Cimento **48 B**, 463 (1967).